Yet after he has finished his demonstration, that straight lines making with the transversal equal alternate angels are parallel, in order to prove the inverse, that parallels cut by a transversal make equal alternate angles, he brings in the unwieldy assumption thus translated by Williamson (Oxford, 1781):

"11. And if a straight line meeting two straight lines make those angles which are inward and upon the same side of it less than two right angles, the two straight lines being produced indefinitely will meet each other on the side where the angles are less than two right angles."

As Staeckel says, "it requires a certain courage to declare such a requirement, alongside the other exceedingly simple assumptions and postulates."

In the brilliant new light given by Bolyai and Lobachevski we now see that Euclid understood the crucial character of the question of parallels.

There are now for us no better proofs of the depth and systematic coherence of Euclid's masterpiece than the very things which, their cause unappreciated, seemed the most noticeable blots on his work.

Sir Henry Savile, in his Praelectiones on Euclid, Oxford, 1621, p. 140, says: "In pulcherrimo Geometriae corpore duo sunt naevi, duae labes . . ." etc., and these two blemishes are the theory of parallels and the doctrine of proportion; the very points in the elements which now arouse our wondering admiration.

But down to our very nineteenth century an ever renewing stream of mathematicians tried to wash away the first of these supposed stains from the most beauteous body of geometry: First, those in which is taken a new definition of parallels. Second, those in which is taken a new axiom different from Euclid's. Third, the largest and most desperate class of attempts, namely those which strive to deduce the theory of parallels from reasonings about the nature of the straight line and plane angle. Hundreds of mathematicians tried at this. All failed. That eminent man Legendre was trying at this, and contiually failing at it, throughout his very long life. Thus the experience of two thousand years went to show that here some assumption was indispensable. Every species of effort was made to avoid or elude it, but without success. From a letter of Gauss we see that in 1799 he was still trying to prove that Euclid's is the only non-contradicting system of geometry, and that it is the system regnant in the external space of our physical experience. The first is false; the second can never be proven.

Yet even in 1831 the acute logician De Morgan accepted and reproduced a wholly fallacious proof of Euclid's assumption, recently republished, Chicago, 1898. A like pseudo-proof published in Crelle's Journal (1834) deceived even our well known Professor W. W. Johnson, who translated and published it in the Analyst (Vol. III, 1876, p. 103), saying, "this demonstration seems to have been generally overlooked by writers of geometrical text-books, though apparently exactly what was needed to put the theory upon a perfectly sound basis."

The most interesting, and perhaps the most extended of such attempted proofs was by the Italian Jesuit Saccheri, born the fifth of September, 1667, who

joined the Society of Jesus at Genoa on the twenty-fourth of March, 1685. He became teacher of grammar in the Jesuit "Collegio di Brera," where the teacher of mathematics was Tommaso Ceva, a brother of the well-known mathematician Giovanni Ceva (1648-1737, who published in 1678 at Milan a work containing the theorem now known by his name.

Saccheri was in close scientific communion with both brothers and received his inspiration from them. He used Ceva's ingenious methods in his first published work, 1693, solutions of six geometric problems proposed by Count Roger Ventimiglia. His attempt at proving the parallel-postulate is his last work.

"Euclid vindicated from every fleck," which received the "Imprimatur" of the inquisition the thirteenth of July, 1733, that of the Provincial of the Jesuits the sixteenth of August, 1733. Saccheri died the twenty-fifth of October, 1733. All preceding attempts were alike in trying to give a direct positive proof of the postulate; all were alike in their assumption open or hidden, conscious or unconscious, of an equivalent postulate.

Saccheri tries a wholly new way, and thus his book marks an epoch. He never doubted the absolute necessary truth of Euclid's postulate, and so he thinks that the two alternatives, possible if it be taken as not true, must each lead to some contradiction, to some absurdity. He tries the reductio ad absurdum. Ninety years later, 1823, Bolyai János reached the astounding conviction that these alternatives lead not to any contradiction but to the "science absolute of space," a generalization of Euclid's universe. In a letter dated the third of November, 1823, written in the Magyar language, and fortunately preserved for us at Maras Vásárhely in Hungary, Bolyai János writes to his father Bolyai Farkas: "I have discovered such magnificent things that I am myself astonished at them. It would be damage eternal if they were lost. When you see them, my father, you yourself will acknowledge it. Now I cannot say more, only so much: *That from nothing I have created another wholly new world.*"

Suppose we take a few steps into this new universe on the path which opened before Saccheri without his ever suspecting whither it led.

1. If two points determine a line it is called a straight.

2. If two straights make with a transversal equal alternate angles, they have a common perpendicular.

3. A piece of a straight is called a sect.

4. If two equal coplanar sects are erected perpendicular to a straight, if they do not meet, then the sect joining their extremities makes equal angles with them and is bisected by a perpendicular erected midway between their feet. [Proved by folding the figure over, along the third perpendicular.]

5. Considering figures where the right angles made by the equal perpendiculars may be said to be not alternate, and where no two perpendiculars to the same straight meet, the equal angles made with the joining sect at the extremities of the two equal perpendiculars are either right angles, acute angles, or obtuse angles. Distinguish the three cases as hypothesis of right, hypothesis of acute, hypothesis of obtuse.

6. According to these three hypotheses respectively, the join of the extremities of the equal perpendiculars is equal to, greater than, or less than the join of their feet. [Saccheri, Prop. III. Translated by Halsted in the American Mathematical Monthly.

7. Inversely, according as the join of the extremities is equal to, or less than, or greater than the join of the feet, the equal angles will be right, or obtuse, or acute. [S. P. IV.]

8. Corollary. In every quadrilateral containing three right angles and one obtuse, or acute, the sides adjacent to this oblique angle are less than the opposite sides, if this angle is obtuse, but greater if it is acute.

9. The hypothesis of right, if even in a single case it is true, always in every case it alone is true. [S. P. V.]

10. Assuming the principle of continuity, and referring only to figures where no two perpendiculars to the same straight meet; The hypothesis of obtuse, if even in a single case it is true, always in every case it alone is true. [S. P. VI.]

11. With like limitation; The hypothesis of acute, if even in a single case it is true, always in every case it alone is true. [S. P. VII.]

12. The sum of the angles of the rectilined triangle is a straight angle in the hypothesis of right, is greater than a straight angle in the hypothesis of obtuse, is less than a straight angle in the hypothesis of acute. [S. P. IX.]

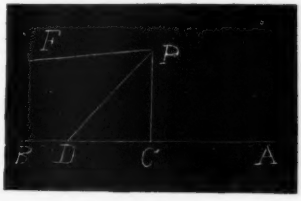
13. The excess of a triangle is the excess of the sum of its angles over a straight angle. The deficiency of a triangle is what its angle-sum lacks of being a straight angle.

14. Two triangles having the same excess or deficiency are equivalent.

15. Even with the assumption that two straights cannot intersect in two points, the three hypotheses give rise to three perfect systems of geometry, the hypothesis of right to Euclid, the hypothesis of acute to Bolyai-Lobachevski, the hypothesis of obtuse to Riemann.

16. In the hypothesis of acute the straight is infinite. Two coplanar straights perpendicular to a third diverge on either side of their common perpendicular. The angle-sum of any rectilineal triangle is less than a straight angle.

17. In Euclid and Bolyai, parallels are straights on a common point at infinity.



18. In Bolyai from any drop point PPC a perpendicular to a given straight AB. If D move off indefinitely on the ray CB the sect PD will approach as limit PF copunctal with AB at infinity. PF is said to be at P the parallel to AB toward B. PF makes with PC an angle CPF which is called the angle of parallelism for the perpendicular PC. It is less than a right angle by an amount which is the limit of the deficiency of the triangle PCD. On the other side of PC an equal angle of parallelism gives us the parallel at P to BA toward A.

Thus at any point there are two parallels to a straight. A straight has two distinct separate points at infinity.

Straights through P which make with PC an angle greater than the angle of parallelism and less than its supplement do not meet the straight AB at all, not even at infinity.

19. A straight maintains its parallelism at all its points. [Lobachevski, Geometrical Researches on the Theory of Parallels, Translated by Halsted, §17.]

20. If one straight is parallel to a second, the second is parallel to the first. [L. §18.]

21. Two straights parallel to a third toward the same part are parallel to each other. [L. §25.]

22. Parallels continually approach each other. [L. §24.]

23. The perpendiculars erected at the middle points of the sides of a triangle are all parallel if two are parallel. [L. §30.]

24. If the foot of a perpendicular slides on a straight, its extremity describes a curve called an equi-distant curve or an equidistantial. An equidistantial will slide on its trace.

25. A circle with infinite radius is not a straight but a curve called the boundary curve, which is a plane curve for which all perpendiculars erected at the mid-points of chords are parallel. [L. §31.] It is an equidistantial whose base line is infinitely removed.

Circles, boundary-curves, equidistantials cut at right angles a system of copunctal straights, of parallel straights, of perpendiculars to a straight, respectively.

Three points determine one of these curves; that is through any three points not costraight will pass either a circle, a boundary-curve, or an equidistantial, and only one such curve.

Any triangle may be inscribed in one and only one of these curves.

26. Boundary-surface we call that surface generated by the revolution of a boundary-curve about one of its axes. Principal plane we call each plane passed through an axis of the boundary-surface.

Every principal plane cuts the boundary-surface in a boundary-curve. Any other plane cuts the boundary surface in a circle.

Boundary-triangles whose sides are arcs of the boundary-curve on the boundary-surface have the same interdependence of angles and sides and the same angle sum as rectilineal triangles in Euclid. Geometry on the boundary surface is the same as the ordinary Euclidean plane geometry. [L. §34.]

27. Triangles on an equidistant-surface are similar to their projections on the base plane; that is, they have the same angles and their sides are proportional.

28. In the hypothesis of obtuse, a straight is of finite size, and returns into itself. This size is the same for all straights. Any two straights can be made to coincide. Two straights always intersect. Two straights perpendicular to a third intersect at a point half a straight from the third either way.

29. A straight in the hypothesis of obtuse does not divide the plane into hemiplanes. Starting from the point of intersection of two straights and passing along one of them over a certain finite sect, we come again to the intersection without having crossed the other straight.

This sect is the whole straight, and so a straight has not really two sides. There is one point through which pass all the coplanar perpendiculars to a given straight. It is called the pole of that straight, and the straight is its polar.

A pole is half a straight from its polar. A polar is the locus of coplanar points half a straight from its pole. Therefore if the pole of one straight lies on another straight, the pole of this second straight is on the first straight.

The cross of two straights is the pole of the join of their poles. The equidistantial is a circle with center at the pole of its basal straight.

Three straights each perpendicular to the other two form a tri-rectangular triangle. It is self-polar, each vertex being the pole of the opposite side.

30. In the hypothesis of obtuse, any two straights enclose a plane figure, a digon. Two digons are congruent if their angles are equal.

31. In the hypothesis of obtuse, all perpendiculars to a plane meet at a point, the pole of the plane. It is the center of a system of spheres of which the plane is a limiting form when the radius becomes equal to half a straight.

Figures on a plane can be projected from similar figures on any sphere which has the pole of the plane for center. They have equal angles and corresponding sides in a constant ratio depending only on the radius of the sphere.

Geometry on a plane is therefore like two-dimensional spherics, but the plane corresponds to only a hemisphere.

The plane is unbounded but not infinite. It is finite in extent. The universe is unbounded but not infinite. It is finite in extent, or content, or volume.

Now of these three possible geometries of uniform space, Euclid's has the unexpected disadvantage that it can never be proved to be the system actual in our external physical world. To establish Euclid, it would be necessary to show that the angle-sum of a triangle is *exactly* a straight angle; and no measurements can ever reach exactitude.

To prove one of the others, we have only to show that the sum of the angles of some triangle is *less than*, or *greater than* a straight angle, which may conceivably be done even by inexact measurements.

What changes ought to be made in teaching elementary geometry in consequence of these later discoveries and the principles of the non-Euclidean geometries?

We are given a new criterion for questions of method, of exposition. For example, surface spherics attains a new importance. When properly founded and expounded, pure spherics, two-dimensional spherics, while giving all the old results and laying the foundation for spherical trigonometry, gives also a picture of the planimetric part of Riemann's geometry, and becomes a touchstone for detecting the fallacies and assumptions in the many pseudo-proofs accepted in the past, such as attempts to found parallelism on direction, attempts to prove all right angles equal, etc.

As another example, we see a new stress laid on the incalculable advantages, educational and scientific, of Euclid's procedure in deducing from three assumed constructions every other construction before he uses it in any demonstration.

The glib method of *supposed* solutions to all desired problems, of hypothetical constructions, is now seen in its deformity and danger. Euclid says, under the heading "Postulates:"

"I. It is assumed, that a straight line may be drawn from any one point to any other point.

"II. And that a terminated straight line [a sect] may be produced in a straight line continually.

"III. And that a circle may be described with any center and radius."

From these Euclid rigidly deduces every problem of construction he wishes to use. Says Helmholtz: "In drawing any subsidiary line for the sake of his demonstration, the well-trained geometer asks always if it is possible to draw such a line. It is notorious that problems of construction play an essential part in the system of geometry.

At first sight these appear to be practical operations, introduced for the training of learners; but in reality they have the force of existential propositions. They declare that points, straight lines, or circles, such as the problem requires to be constructed, are possible under all conditions, or they determine any exceptions that there may be."

Euclid's first three propositions are problems.

The most popular American geometry, Wentworth's, (1899), puts Euclid's two first postulates on page 8, and the third postulate a whole book later, and then never has a single problem of construction until page 112, where he says: "Hitherto we have supposed the figures constructed."

Meantime, on page 88, he gives as a "theorem:" "Through three points not in a straight line one circumference, and only one, can be drawn."

He gives as his "Proof. Draw the chords AB and BC. At the middle points of AB and BC suppose perpendiculars erected. These perpendiculars will intersect at some point O, since AB and BC are not in the same straight line."

Now the tremendous existential import of the problem, to draw a circle through three non-costraight points, will be recognized when I say that in general it is not possible. In the Lobachevski geometry not every triangle has its vertices concyclic. Granting that every three points must be costraight or concyclic, we could prove the parallel-postulate.

Of the possible geometries we cannot say *a priori* which shall be that of our actual space, the space in which we move.

The hereditary geometry, the Euclidean, is underivable from real experience alone, and can never be proved by experience. Euclidean space is, at least in part, a creation of the human mind. Its adequacy as a subjective form for experience has not yet been disproved.

It can never be proved.

The realities which with the aid of our subjective space form we understand under motion and position, may, with the coming of more accurate experience refuse to fit in that form. Our mathematical reason may decide that they would be fitted better by a non-Euclidean space form.

Comparative geometry finally overthrows that superficial method which pretends to found a logically sound exposition of geometry on "direction," undefined.

For more than twenty years Wentworth gave as his definition "A straight line is a line which has the same direction throughout its whole extent." [1877, Def. 8. 1886, p. 4; 1888, §17.]

At last he discards his aged error, and takes the definition of non-Euclidean geometry, "a straight is the line determined by two points." [1899, §§36 and 46.]

Though the Bolyai and the Riemann geometries are founded on the straight, yet to say in them of two straights that they have the same direction has no ordinary meaning, since in Riemann every two straights cross and inclose a space, while in Bolyai every two parallels continually approach each other. So as to direction, Wentworth has reformed, after twenty years in the land of Nod. But he still says, 1899, §49: "A straight line is the shortest line that can be drawn from one point to another."

Now a relation of equality or inequality between two magnitudes must have some foundation, and be capable of some intelligible test. In the traditional geometry the foundation of all proof by Euclid's method consists in establishing the congruence of magnitudes. To make the congruence evident, the geometrical figures are supposed to be applied to one another, of course without changing their form and dimensions. But since no part of a curve can be congruent to any piece of a straight, so, for example, no part of a circle can be equivalent to any sect from the definition of equivalent magnitudes as those which can be cut into pieces congruent in pairs.

In any comparison of size by congruence, we must be able to place one of the magnitudes or portions of it in complete or partial coincidence with the other. No such direct comparison can be instituted between a straight and a line no piece of which is straight.

Thus the whole of Euclid's *Elements* fails utterly to institute or prove any relation as regards size between a sect and an arc joining the same two points. The operation of measurement we cannot effect, rigorously speaking, either for curves or for curved surfaces, since the unit for length is a sect, and the unit for area, the square on that sect. In fact, however little may be the parts of a curve, they do not cease to be curved, and consequently they cannot be compared directly with a sect; just as parts of a curved surface are not directly comparable with portions of a plane.

We cannot even affirm that any ratio exits between a circle and its diameter until after we have made some extra-Euclidean and post-Euclidean assumption at least equivalent to the following:

No minor arc is less than its chord; and no arc is greater than the sum of

the tangents at its extremities. If the curve be other than a circle we assume that on it one can always take two points so near that the arc between these points is not less than its chord, nor greater than the broken line formed by the two tangents touching its extremities. Some such assumption is, in fact, necessary, but it destroys by itself the primitive idea of measuring curves with straights.

Duhamel gives the assumption the following form: The length of a curve shall be the limit toward which the length of a broken line made up of consecutive chords of that curve approaches, when the number of chords is increased in such a manner that the chords all approach zero as a limit. Thus the elevation of the length of a curve represents not at all an attempt at rectification strictly; but it has for aim the finding of a limit to which another magnitude would approach.

In geometry one proves that as the subdivisions are increased and the sides tend toward the limit zero, the perimeter of the polygon inscribed in a circle increases, circumscribed decreases, toward the same limit, which then is assumed for the magnitude of the circle.

Therefore when Phillips and Fisher, of Yale, give as their definition of a straight [1898, p. 4, §7. Def.] "A *straight line* is a line which is the shortest path between any two of its points," they pass through and beyond Euclid's *Elements* to give us his simplest element; they institute a comparison not only with circular arcs, but also with all curves known and unknown; they presuppose a foreknowledge of all lines in a definition of the simplest line. Is it still needful to say this is grossly bad logic, bad pedagogy, bad mathematics?

The same Yale geometry blunders strikingly on p. 23, where it says: "In fact, Lobatchewsky in 1829, proved that we can never get rid of the parallel axiom without assuming the space in which we live to be very different from *what we know it to be through experience.* Lobatschewsky tried to imagine a different sort of universe in which the parallel axiom would not be true. This imaginary kind of space is called *non-Euclidean* space, *whereas the space in which we really live* is called *Euclidean*, because Euclid (about 300 B. C.) first wrote a systematic geometry of *our space.*"

The scientific doctrine of evolution postulates a world independent of man, and teaches the outcome of man from lower forms of life in accordance with wholly natural causes. In this world of evolution experience is a teacher, but man is a creator, and the mighty examiner is death.

The puppy born blind must still be able, guided by the sense of smell, to superimpose his mouth upon a source of nourishment. The little chick, responding to the stimulus of a small bright object, must be able to bring his beak into contact with the object so as to grasp and then swallow it. The springing goat that misjudges an abyss is lost.

So too with man. His ideas must in some way correspond to this independent world, or death passes upon him an adverse judgment. But it is of the very essence of the doctrince of evolution that man's metric knowledge of this

independent world, having come by gradual betterment and through imperfect instruments, for example the eye, cannot be absolute and exact.

The results of any observations are always with certain definite limitations as to exactitude and under particular conditions. Man the creator replaces these results by assumptions presumed to have absolute precision and generality, such as, for example, the so-called axioms of Euclid.

If two natural hard objects, susceptible of high polish, be ground together, their surfaces in contact may be so smoothed as to fit closely together and slide one on the other without separating. If now a third surface be ground alternately against each of these two smooth surfaces until it accurately fits both, then we say that each of the three surfaces is approximately plane, is a piece of a plane. If one such plane be made to cut through another, we say the common line where they cross is approximately a straight. The perfect, the ideal plane, is a human creation under which we seize the imperfect data of experience.

If three approximate planes on real objects be made to cut through a fourth approximate plane, then three approximate straights are formed on this fourth plane, and in general they are found to intersect, and the figure they make we may call an approximate triangle. Such triangles vary greatly in shape. But no matter what the shape, if we cut off the six ends of any two such, and place them side by side on a plane with their verticles at the same point, the six are found with a high degree or approximation just to fill up the plane about the point. If the whole angular magnitude about any point in a plane be called a perigon, then we may say that the six angles of any two approximate triangles are found to be together approximately a perigon.

Now does the exactness of this approximation to a perigon depend only on the straightness of the sides of the original two triangulars, or also upon their size?

If we know with absolute certitude that the size of the triangles has nothing to do with it, then we know something that we have no right to know according to the doctrine of evolution, something impossible for us ever to have learned evolutionally.

Yet before the epoch-making ideas of Bolyai János and Lobachevski, every one supposed we were perfectly sure that the angle-sum of an actual approximate triangle approached a straight angle with an exactness dependent only on the straightness of the sides and not at all on the size of the triangle. But if in the mechanics of the world independent of man we were absolutely certain that all therein is Euclidean and only Euclidean, then Darwinism would be disproved by the reductio ad absurdum.

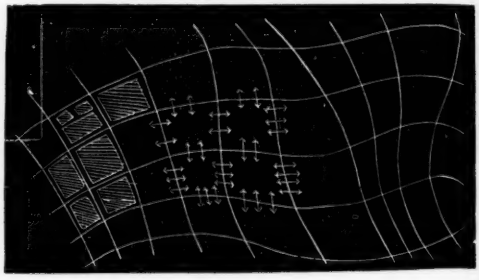
All our measurements are finite and approximate only. The mechanics of actual bodies in what Cayley called the external space of our experience, might conceivably be shown by merely approximate measurements to be non-Euclidean, just as a body might be shown to weigh more than two grams or less than two grams, though it never could be shown to weigh precisely, absolutely two grams.

The outcome of the non-Euclidean geometry is a new freedom to explain and understand our universe and ourselves.

ON THE PROJECTIVITY OF STRESSES IN A PLANE.

By ARNOLD EMCH, Ph. D., The University of Colorado.

1. For geometrical purposes we may assume that a solid having the shape of a slab and subjected to stresses parallel to its forces reduces to a plane surface of infinitesimal thickness. If we now consider any point A of the plane, and the pencil of rays passing through this point, every ray represents a section which is subject to definite strains. Let s be such a ray and $\triangle s$ a small line-element of s at the point A, and $\triangle R$ the resultant of the strains acting upon $\triangle s$; then $\frac{\triangle s}{\triangle R}$ is called the *specific stress* in the line element $\triangle s$ and is evidently also expressed by



$$\frac{ds}{dR}=\lim\left(\frac{\triangle s}{\triangle R}\right)=\rho, \text{ say.}$$

In general the quantity ρ is different for every section and may assume a maximum and minimum at a point A. The laws of specific stresses of all sections through A may easily be obtained by the methods of graphic statics* and are as follows:

(*a*). *If ρ is the specific stress acting at A upon a section S, then the specific stress ρ' acting upon a section S' which is parallel to ρ, is parallel to S; i. e., if S' is parallel to the direction of ρ, then ρ' is parallel to the direction of S.*

(*b*). *All pairs of rays S and S' defined by (a) form an involution of rays. As each involution contains a rectangular pair, r perpendicular to r', it follows that at each point A there are two perpendicular sections each of which is perpendicular to its corresponding specific stress.*

(*c*). *If the involution has double rays, it is evident that there are only shearing stresses along these sections. These rays separate the sections in which the material is only subjected to tensions from those subjected to compression only. If one ray S is subjected to tension, then the corresponding ray S' is subjected to compression only. The same is true of the rectangular pair (r, r').*

(*d*). *If the double rays of the involution are imaginary, then all sections are snbjected to only one kind of stresses, either tension or compression. In this case there are two perpendicular sections at A which are subjected to normal stresses of the same kind.*

2. In both kinds of involution it is now possible to construct two systems of curves in a plane, so that all tangents of these curves, representing sections through the material, are perpendicular to their corresponding stresses.

*See Ritter, *Anwenendungen Graphischen Statik*, Vol. I, pages 1—25, Zurich.

In the case of an hyperbolic involution (*c*) these curves consist of two orthogonal systems of curves of normal compressions and tensions. If the involution is elliptic the orthogonal systems consist of curves whose tangents are all subjected to the same kind of normal stresses, *i. e.*, either to tension or to compression only. The first case is well known in graphic statics. Little attention has, however, been paid to the elliptic involution of stresses in a plane, and it will be interesting to illustrate this case by a few examples which strikingly exhibit the character of orthogonality of the curves of normal tensions.

If the material of a slab is only subjected to external tensions, the curves of normal tensions will resemble those of the adjoining figure.

As soon as the stresses in certain portions of the material exceed the strength of the material, the rupture of the material will take place along the curves of normal tensions, as it is evident from the figure.

This fact is beautifully illustrated by the cracks that form in a drying mass of mud along a river, where by the contraction of the mass only tensions are produced. It may also be observed on a heavily varnished surface, and in numerous other examples which depend only on tension.

In a future article the author intends to find orthogonal systems of normal stresses for a number of cases where the law of the distribution of specific stresses in a plane is a given function of the coördinates and of the angle of inclination of the line-element at a point.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

126. Proposed by G. B. M. ZERR, A.M., Ph.D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Bought 150 head of stock for $300, paying for each kind $2 5-6, $1 5-9, and $5-7, respectively. Find number of each kind bought.

I. Solution by MARCUS BAKER, U. S. Coast and Geodetic Survey, Washington, D. C; H. C. WHITAKER, Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pa.; COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.; and CHARLES C. CROSS, Meredithville, Va.

The conditions are $x+y+z=150$, $2\frac{5}{6}x+1\frac{5}{9}y+\frac{5}{4}z=300$, and these to be solved in positive integers.

Eliminating x we have $2\frac{3}{18}y+\frac{8}{4}\frac{9}{2}z=125$; or

$$y=97-z+\frac{133-106z}{161}. \quad \text{Put } 133-106z=161a.$$

$\therefore\ z=1-a+\frac{27-55a}{106}$. Put $27-55a=106b$.

$\therefore\ a=-b+\frac{3(9-17b)}{55}$. Put $9-17b=55c$.

$\therefore\ b=-3c+\frac{9-4c}{17}$. Put $9-4c=17d$.

$\therefore\ c=2-4d+\frac{1-d}{4}$. Put $1-d=4n$.

$\therefore\ d=1-4n$; and by substituting through the various steps,

$c=17n-2$,
$b=7-55n$,
$a=106n-13$.

$z=21-161n$
$y=63+267n$
$x=66-106n$

Here n may have *any* value, but $n=0$ alone gives positive integral values for x, y and z.

Therefore he bought of the

First kind,	66 at $\$2\frac{5}{6}$,	$187
Second kind,	63 at $\$1\frac{5}{9}$,	98
Third kind,	21 at $\$\ \frac{5}{7}$,	15
	150	$300

II. Solution by S. F. NORRIS, Professor of Astronomy and Mathematics, Baltimore, Md.; SYLVESTER ROBBINS, North Branch, N. J.; P. S. BERG, B. S., Principal of Schools, Larimore, N. D.; M. A. GRUBER, A. M., War Department, Washington, D. C.; and ARCHIE C. MURRY, Baltimore, Md.

Average price $=\$2.\ \left\{\begin{matrix}2\frac{5}{6}\\1\frac{5}{9}\\\frac{5}{7}\end{matrix}\right\}$

	1st	2nd		Res't
	8	54	$33\frac{3}{5}$	66
15		63		63
	35		21	21

Balancing gains and losses:

On 1 animal at $\$2\frac{5}{6}$, lose $\$\frac{5}{6}$; hence buy 8 animals at $\$2\frac{5}{6}$.
On 1 animal at $\$1\frac{5}{9}$, gain $\$\frac{4}{9}$; hence buy 15 animals at $\$1\frac{5}{9}$. } 1st.

On 1 animal at $\$2\frac{5}{6}$, lose $\$\frac{5}{6}$; hence buy 54 animals at $\$2\frac{5}{6}$.
On 1 animal at $\$\frac{5}{7}$, gain $\$\frac{2}{7}$; hence buy 35 animals at $\$\frac{5}{7}$. } 2nd.

NOTE.—To get required number, multiply 1st column by $4\frac{1}{5}$, and 2nd column by $\frac{3}{5}$.

Answer.	66 animals at $\$2\frac{5}{6}$ =	$187
	63 animals at $1\frac{5}{9}$ =	98
	21 animals at $\frac{5}{7}$ =	15
	150 animals for	$300

ALGEBRA.

103. Proposed by WALTER H. DRANE, A. M., Graduate Student, Harvard University, Cambridge, Mass.

Given the equation $x^m+p_1x^{m-2}+p_2x^{m-2}+\ldots.p_{m-1}x+p_m=0$ freed from multiple roots. Prove that its discriminant is positive or negative according as the number of pairs of complex roots is even or odd.

I. Solution by the PROPOSER.

We have $\triangle=(x_1-x_2)^2(x_1-x_3)^2(x_1-x_4)^2\ldots.(x_1-x_m)^2$
$(x_2-x_3)^2(x_2-x_4)^2\ldots.(x_2-x_m)^2$
$(x_3-x_4)^2\ldots.(x_3-x_m)^2$
$\ldots\ldots\ldots\ldots\ldots$
$(x_{m-1}-x_m)^2.$

There are seven cases to be considered:

I. Differences of real roots.
II. Differences of a real and a complex root.
III. Differences of a real and a pure imaginary root.
IV. Differences of two pure imaginaries.
V. Differences of a pure imaginary and a complex.
VI. Differences of two complex roots not conjugate.
VII. Differences of two conjugate complex roots.

I. It is evident at once, since each difference is squared, that the real roots alone can not affect the sign.

II. If c be a real root, and $a+bi$ a complex, for every difference of the form $c-(a+bi)$ we shall also have one of the form $c-(a-bi)$ and as the product of these two is positive, their square is also positive, and hence case II can not affect the sign of $\triangle$.

By exactly the same reasoning it can be shown that cases III, V, and VI cannot affect the sign of $\triangle$.

IV is a special form of VII, so we shall pass to the last. Let $a+bi$ and $a-bi$ be two conjugate complex roots. Then the difference is $\pm bi$ whose square is $-b$. Hence for every pair of roots we use in forming a difference, we change the sign of $\triangle$, and hence $\triangle$ is positive or negative according as the number of pairs of complex roots is even or odd.

II. Solution by J. W. YOUNG, Fellow and Assistant in Mathemetics, Ohio State University, Columbus, O.

The discriminant is equal to the product of squares of the differences of roots taken in all possible combinations, *i. e.*: $\triangle=\Pi(x_1-x_2)^2$ [x, x_2, $x_3\ldots x_n$ =roots of equation].

For every complex pair of roots, there will be a factor of the form

$$[(\xi+i\eta)-(\xi-i\eta)]^2=[2i\eta]^2=-4\eta^2.$$

That is, for every complex pair there will be a *negative* factor in the discriminant.

Moreover the remaining factors will give you a *positive* product. For, if one of the factors be

$$[(\xi_1+i\eta_1)-(\xi_2+i\eta)]^2=[(\xi_1-\xi_2)+i(\eta_1-\eta_2)]^2$$

there will be a corresponding factor

$$[(\xi_1-i\eta_1)-(\xi_2-i\eta_2)]^2=[(\xi_1-\xi_2)-\mathrm{i}(\eta_1-\eta_2)]^2.$$

When multiplied these give

$$[(\xi_1-\xi_2)^2+(\eta_1-\eta_2)^2]^2,$$

which is clearly positive. So all the factors containing complex roots may be taken in pairs which give a positive product, and, of course, the factors containing real roots only are positive. Hence corresponding to every pair of complex roots there will be *one* and *only one* negative factor. Therefore, if the number of pairs be even the discriminant will be positive; if odd, negative.

NOTE.—Mr. Harry S. Vandiver should have been credited as a joint author of the solution of problem 102.

GEOMETRY.

129. Proposed by **WILLIAM HOOVER, A. M., Ph. D.**, Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Show that at no point of an ellipse will the circle of curvature pass through the center, if the eccentricity be less than $\frac{1}{2}\sqrt{2}$.

I. Solution by **F. ANDEREGG, A.M.**, Professor of Mathematics, Oberlin College, Oberlin, O.; **W.H. CARTER, A. M.**, Professor of Mathematics, Centenary College, Jackson, La.

Since for the ellipse the radius of curvature is

$$\rho=\frac{(a^4y^2+b^4x^2)^{\frac{3}{2}}}{a^4b^4};$$

and the center of curvature is the point

$$\left(\frac{(a^2-b^2)x^3}{a^4},-\frac{(a^2-b^2)y^3}{b^4}\right),$$

the equation of the circle of curvature is

$$\left(x'-\frac{(a^2-b^2)x^3}{a^4}\right)^2+\left(y'+\frac{(a^2-b^2)y^3}{b^4}\right)^2=\frac{(a^4y^2+b^4x^2)^3}{a^8b^8}.$$

This circle passes through the origin

$$(a^2-b^2)^2(b^8x^6+a^8y^6)=(a^4y^2+b^4x^2)^3.$$

This equation is easily simplified, and assumes the form

$$\frac{x^2}{a^2}=\frac{1+e^2}{3e^2}.$$

It follows at once that the least value of $3e^2$ is $1+e^2$, or the least value of e is $\frac{1}{2}\sqrt{2}$.

II. Solution by the PROPOSER.

If ϕ be the eccentric angle of any point of the ellipse $a^2y^2+b^2x^2=a^2b^2$, the equation to the corresponding circle of curvature is

$$x^2+y^2-(a^2-b^2)\left(\frac{2\cos^3\phi}{a}x-\frac{2\sin^3\varphi}{b}y\right)$$

$$+a^2(\cos^2\varphi-2\sin^2\varphi)-b^2(2\cos^2\varphi-\sin^2\varphi)=0.$$

This passing through the center, requires that

$$a^2(\cos^2\phi-2\sin^2\varphi)-b^2(2\cos^2\varphi-\sin^2\varphi)=0,$$

$$\text{or } \frac{a^2-b^2}{a^2}=e^2=\frac{1}{2-3\sin^2\varphi},$$

which is a minimum for $\phi=0$ or π; that is $e^2=\frac{1}{2}$.

Excellent solutions were received from *J. W. YOUNG*, *G. B. M. ZERR*, *J. SCHEFFER*, and *W. H. DRANE*.

CALCULUS.

98. Proposed by CHARLES CARROLL CROSS, Meredithville, Va.

Of the circumference of a fixed circle radius R rolls a circle radius r. Required the length of the curve described by a point on the circumference of the rolling circle; (1) when the circle rolls on the inside; (2) when the circle rolls on the outside of the circumference of the fixed circle.

Solution by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.; WALTER H. DRANE, A. M., Graduate Student, Harvard University, Cambridge, Mass.; J. SCHEFFER, A. M., Hagerstown, Md.; M. E. GRABER, Student, Heidelberg University, Tiffin, O.; and G. B. M. ZERR, A.M., Ph.D., Professor of Science and Mathematics, Chester High School, Chester, Pa.

We have here the epicycloid and hypocycloid. The equation of the former is

$$x=(R+r)\cos\phi-r\cos\frac{R+r}{r}\phi, \text{ and } y=(R+r)\sin\phi-r\sin\frac{R+r}{r}\phi.$$

$$\therefore \frac{dx}{d\phi}=-(R+r)\sin\phi+(R+r)\sin\frac{R+r}{r}\phi.$$

$$\frac{dy}{d\phi}(R+r)\cos\phi-(R+r)\cos\frac{R+r}{r}\phi.$$

But $\frac{ds^2}{d\phi^2}=\frac{dx^2}{d\phi}+\frac{dy^2}{d\phi^2}=4(R+r)^2\sin^2\frac{R}{2r}\phi.$

$\therefore$ Length of curve between cusp and cusp

$$=2\int_0^{2+r/R}(R+r)\sin\frac{R}{2r}\phi=\frac{8r}{R}(R+r).$$

In the case of the hypocycloid we have in its equation only to put $-r$ in lieu of r, and by proceeding in the same way we obtain the length of the curve

$$=\frac{8r}{R}(R-r).$$

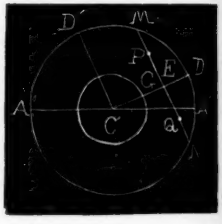
AVERAGE AND PROBABILITY.

86. Proposed by L. C. WALKER, Assistant in Mathematics in Leland Stanford, Jr., University, Palo Alto, Cal.

Two points are taken at random in a circular annulus formed by two concentric circles. Find the chance that the straight line joining the points will not cut the inner variable circle.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let P, Q be the two random points, MN the chord through P, Q



Let $AC=r$, $CE=w$, $MQ=x$, $PQ=y$, $\angle ACD'=\theta$. CG, the radius of the variable circle $=u$, $p=$required chance.

An element of the circle at Q is $dwdx$, at P $ydθdy$. The limits of u are 0 and r; of w, r and u; of x, $2\sqrt{(r^2-w^2)}$ and 0; of y, 0 and x and doubled; of θ, 0 and 2π.

$$\therefore\ p=\frac{\frac{2}{\pi^2r^4}\int_0^r\int_u^r\int_0^{2\sqrt{(r^2-w^2)}}\int_0^x\int_0^{2\pi}dudwdxydyd\theta}{\int_0^r du}$$

$$=\frac{4}{\pi r^5}\int_0^r\int_u^r\int_0^{2\sqrt{(r^2-w^2)}}\int_0^x dudwdxydy$$

$$= \frac{2}{\pi r^5}\int_0^r \int_u^r \int_0^{2\sqrt{(r^2-w^2)}} x^2\,dudwdx$$

$$= \frac{16}{3\pi r^5}\int_0^r \int_u^r (r^2-w^2)^{\frac{3}{2}}\,dudw$$

$$= \frac{1}{3\pi r^5}\int_0^r [3\pi r^4 - 10r^2u\sqrt{(r^2-u^2)} + 4u^3\sqrt{(r^2-u^2)} - 6r^4\sin^{-1}\frac{u}{r}]du$$

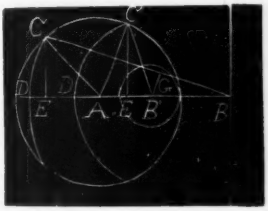
$$= \frac{16}{15\pi} = .3395.$$

87. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Science and Mathematics, Chester High School, Chester, Pa.

Find the mean distance of a random point in a sphere from a point, (1) within, (2) without the sphere.

Solution by the PROPOSER.

Let B be the given point. $AB=c$, $AC=a$, $AD=\pm x$.



Then $DB=c-x=BC$, $CE=\sqrt{(a^2-AE^2)}$
$=\sqrt{[(c-x)^2-(c-AE)^2]}$.

$\therefore\ AE=\dfrac{(a^2+2cx-x^2)}{2c}$,

$$DE=AE-x=\frac{(a^2-x^2)}{2c}.$$

Area of film $CDM=2\pi DB.DE=2\pi(c-x)(a^2-x^2)/2c$.

Also let $BG=y$. Then the area of the surface of this sphere is $4\pi y^2$.

(1). Distance $D=\dfrac{\int_{-a}^{2c-a} 2\pi DB^2.DEdx + \int_0^{a-c} 4\pi y^3 dy}{\frac{4}{3}\pi a^3}$

$$\therefore\ D=\frac{3}{4a^3c}\int_{-a}^{2c-a}(c-x)^2(a^2-x^2)dx+\frac{3}{a^3}\int_0^{a-c}y^3dy.$$

$$\therefore\ D=\tfrac{3}{4}a+\frac{c^2}{2a}-\frac{c^4}{20a^3}.$$

(2). $D_1=\dfrac{\int_{-a}^{a} 2\pi DB^2.DEdx}{\frac{4}{3}\pi a^3}=\dfrac{3}{4a^3c}\int_{-a}^{a}(c-x)^2(a^2-x^2)dx.$

$$\therefore\ D_1=c+(a^2/5c).$$

MISCELLANEOUS.

77. Proposed by T. E. COLE, Columbus, Ohio.

It is said that a base-ball pitcher throws curves. Give a scientific explanation of how it is done.

Solution by WALTER H. DRANE, A. M., Graduate Student, Harvard University, Cambridge, Mass.

When a ball pitcher wishes to throw a curve, in addition to the onward motion given, he sets the ball to revolving rapidly about its own axis. Now the resistance the air offers to a moving body depends upon its velocity, and the only resistance besides gravity which the ball encounters in its motion is this resistance of the air. Consider the ball as revolving from right to left as in the figure, and let us regard it as two bodies, the line of division being the line of its own motion onward. The particles of the right half of the ball are moving around either directly in the same direction as the onward motion, or in oblique directions; but these oblique velocities may all be resolved into two velocities. One parallel to and in the same direction OF, and one perpendicular to OF, which last has no effect on the onward velocity. Consequently the particles of the right half have, in addition to the onward velocity of the ball, a velocity due to their revolution about O, this increase of velocity depending upon the distance of a particle from the center. Now in exactly the same way it can be shown that the velocity of particles on the left is decreased in proportion as that of those on the right is increased. Hence the resistance of the air offered to particles on the sight is slightly greater than that offered on the left. The result is that in effect a backward force is brought into play on the right which does not act on the left. Resolve this force parallel and perpendicular to OF. That parallel to OF does not effect the direction of motion. But combining that component perpendicular to F with F itself, we get that the total effect is to change the direction of the onward velocity, and since both the onward velocity and the revolution of the ball about its axis change at every instant of time, this change in direction must vary at any instant, and hence the ball moves in a curved line to which OR is a tangent.

This is merely a case of constrained motion as the above shows.

Also solved by *ALOIS F. KOVARIK*.

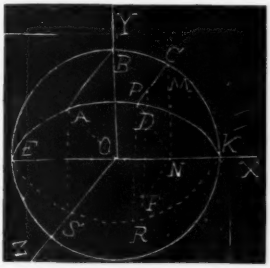
78. Proposed by WALTER H. DRANE, A. M., Graduate Student, Harvard University, Cambridge, Mass.

The center of a regular polygon of n sides moves along a diameter of a given circle, the plane of the polygon being perpendicular to the diameter, and its magnitude varying in such a manner that one of its diagonals always coincides with a chord of the circle; find the surface and the volume generated, and thence deduce the formulae for the surface and the volume of a sphere.

Solution by the PROPOSER.

Let AB be a side of the polygon in an initial position; CD the side after

it has moved an infinitesimal distance ON. Then $ABCD$ is an elementary portion of $1/n$th the required area, and the prism $AOB-DNC$ an elementary portion of $1/n$th the required volume. The equation of the plane $ADKO$ is $z=\cot(\pi/n).y$. Let the equation of the sphere of which the circle KBE is a great circle be



$$x^2+y^2+z^2=a^2.$$

Combining the equation of the plane and sphere gives $x^2+\sec^2(\pi/n)z^2=a^2$, which is the equation of $ESRK$, the projection on the plane xz of the curve of intersection of the plane and sphere. Now let P be any point in the area. $CN=\sqrt{(a^2-x^2)}$. $CM=MP\tan MPC=MP\tan(\pi/n)=z\tan(\pi/n)$.

$PF=y$; $CN=CM+PF$ or $\sqrt{(a^2-x^2)}=z\tan(\pi/n)+y$, which is the equation of the surface.

$$\frac{A}{2n}=\int\frac{\sqrt{\left[\left(\frac{df}{dx}\right)^2+\left(\frac{df}{dy}\right)^2+\left(\frac{df}{dz}\right)^2\right]}}{\frac{df}{dy}}$$

$$=\int_0^a\int_0^{\cos(\pi/n)\sqrt{(a^2-x^2)}}\frac{\sqrt{[a^2\sec^2(\pi/n)-x^2\tan^2(\pi/n)]}}{\sqrt{[a^2-x^2]}}\,dz\,dx$$

$$\cos(\pi/n)\int_0^a\sqrt{[a^2\sec^2(\pi/n)-x^2\tan^2(\pi/n)]}\,dx$$

$$=\frac{\sin(\pi/n)}{2}\left[a^2\cot(\pi/n)+(\pi a^2/n)\operatorname{cosec}^2(\pi/n)\right].$$

$$\therefore\ A=na^2\cos(\pi/n)+\frac{\pi a^2}{\sin(\pi/n)}.$$

Evaluating this when $n=\infty$, by reducing to a common denominator, and then differentiating both numerator and denominator four times, this becomes $4\pi a^2$, the area of a sphere.

For volume we have,

$$V/2\pi=\sin(\pi/n)\cos(\pi/n)\int_0^a(a^2-x^2)\,dx=\tfrac{2}{3}a^3\sin(\pi/n)\cos(\pi/n).$$

$$\therefore\ V=\tfrac{4}{3}na^3\sin(\pi/n)\cos(\pi/n)=\tfrac{2}{3}na^3\sin(2\pi/n).$$

Throwing this in the typical form, and evaluating when $n=\infty$ by differentiating twice, gives $\frac{4}{3}\pi a^3$, the volume of a sphere.

Also solved by G. B. M. ZERR.

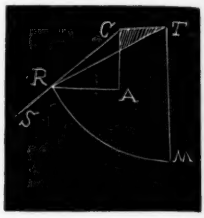
79. Proposed by S. HART WRIGHT, M. D., A. M., Ph. D., Penn Yan, N. Y.

In latitude 42° 30′ N.$=\lambda$, a tree 100 feet long$=\alpha$, leans in the direction S. 60° W.$=\beta$, with an angle of elevation with the level ground, of 30°$=\gamma$. The sun's declination being 1° 36′ 24″ N.$=\delta$, in what direction will the shadow of the tree point, when the sun is on the meridian ?

I. Solution by the PROPOSER.

Let $TR=100=a$, be the tree, the top at R, TM a south line from T.

Draw a radii from R to A, and AT will be a sub-projection of RT, on the ground, the triangles TAR and CAR being vertical. Angles MTR, and $MTA=60°$ each$=\beta$, the triangle CAT being horizontal. Angle $ATR=30°=\gamma$, being the elevation of TR.



The shadow of R made by the sun at S must be cast north of A, and appear on the ground at C, and all the termini of shadows of all other points from R to T must be cast and visible on the line CT, as the sun is practically south of all points on the tree.

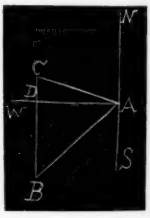
Angle $ACR=90°-\lambda+\delta=49°\ 6'\ 24''=c$=the sun's meridian altitude. $AR=a\sin\gamma=50$ feet$=e$, $AC=e\cot c=43.3013$ feet$=\delta$, $AT=a\cos\gamma=86.6025$ feet$=b$.

Whatever angle CT makes with TM, counted from the south westward, will give the direction of the shadow.

CA must be parallel with TM. $\therefore$ Angle $CAT=ATM=60°$. Then in triangle ACT we have given AC and AT, and the included angle of 60°, to find the angle CTA, which, by Plane Trigonometry, is easily found to be 30°, ACT becoming in this particular case a right angle. Adding the angles CTA and ATM gives $MTC=90°$, and therefore the shadow points west.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Let NS be a north and south line on the ground, AB the projection of the tree upon the ground, AW the west line on the ground, BL perpendicular to AW, AC the shadow, $\angle NAC=\alpha$, and $\angle BAS=\beta$. $AB=a\cos\gamma$, $BD=a\cos\gamma\cos\beta$.



Since $90°-\lambda+\delta$ is the meridianal altitude we have $BC=a\sin\gamma\tan(\lambda-\delta)$.

$\therefore$ $CD=a\sin\gamma\tan(\lambda-\delta)-a\cos\gamma\cos\beta$.

$$\therefore\ \cot x=\frac{CD}{AD}=\frac{a\sin\gamma\tan(\lambda-\delta)-a\cos\gamma\cos\beta}{a\cos\gamma\sin\beta}.$$

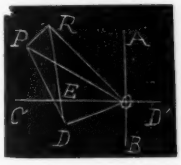
$$\therefore\ \cot x=\frac{\tan\gamma\tan(\lambda-\delta)}{\sin\beta}-\cot\beta.$$

Putting $\dfrac{\tan\gamma\tan(\lambda-\delta)}{\sin\beta}=\cot\varepsilon$, we get $\cot x=\dfrac{\sin(\beta-\varepsilon)}{\sin\beta\sin\varepsilon}$.

For the given numerical values, $\gamma=30^\circ$, $\beta=60^\circ$, $\lambda-\delta=40^\circ\ 53'\ 36''$, we get $\varepsilon=60^\circ$. $\therefore\ \beta-\varepsilon=60^\circ$. $\therefore\ \cot x=0$. $\therefore\ x=90$. Consequently the shadow falls due west.

III. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let $OP=100$ feet$=a$ be the tree, AB the meridian, CD' the parallel of latitude, OD the projection of the tree on the plane, OR the shadow, $\angle POD=30^\circ=\gamma$, $\angle DOB=60^\circ=\beta$, $\angle PRD=$sun's altitude$=\frac{1}{2}\pi-(\lambda-\delta)$, $\angle RDO=\beta$, $\angle DOE=\frac{1}{2}\pi-\beta$, $\angle PDO=\angle PDR=\angle DEO=\frac{1}{2}\pi$.



$\therefore\ PD=a\sin\gamma=50$ feet, $DO=a\cos\gamma=50\sqrt{3}$.

$\therefore\ DO=86.60$ feet, $DE=DO\cos\beta=43.30$ feet, $DR=PD\tan(\lambda-\delta)=a\sin\gamma\tan(\lambda-\delta)=43.30$ feet. $\therefore\ DE=DR$ and E and R coincide.

$\therefore$ The shadow is due west.

Also solved by *EDMUND FISH* and *A. H. BELL*, and *H. C. WHITAKER*.

80. Proposed by the late SYLVESTER ROBINS, North Branch, N. J.

Exhibit ten initials in that infinite series of integral, rational rhombuses wherein the area of every term is one unit less than the square of its side.

Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

Let $2a$ and $2b$=the respective diagonals of a rhombus. As the diagonals of a rhombus bisect each other at right angles, we then have, side of rhombus$=\sqrt{(a^2+b^2)}$, and area$=2ab$.

$\therefore$ From condition of problem, $2ab=a^2+b^2-1$, or $a^2-2ab+b^2=1$.

Whence $a-b=\pm 1$.

$\therefore$ *The side of rhombus must be the hypothenuse of a right triangle whose legs are consecutive integers.*

Several methods for finding successive right triangles of this kind are given in THE AMERICAN MATHEMATICAL MONTHLY, Vol. IV, No. 1, pages 24—27.

Whence we find, for the first ten integral, rational rhombuses, the respective—

Diagonals,		*Sides*,	and *Areas*.
$2a$	$2b$	$\sqrt{(a^2+b^2)}$	$2ab=a^2+b^2-1$
8	6	5	24
42	40	29	840
240	238	169	28560
1394	1392	985	970224
8120	8118	5741	32959080
47322	47320	33461	1119638520
275808	275806	195025	38034750624
1607522	1607520	1136689	1292061882720
9369320	9369318	6625109	43892069261880
54608394	54608392	38613965	1491038293021224

The general formula for finding sides is $6S_{n-1}-S_{n-2}=S_n$.

Also solved by *A. H. BELL*, *CHAS. C. CROSS*, and *G. B. M. ZERR*.

PROBLEMS FOR SOLUTION.

ALGEBRA.

119. Proposed by HARRY S. VANDIVER, Bala, Montgomery Co., Pa.

Given $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{3\times5} + \frac{17x^7}{3^2\times5\times7} + \frac{62x^9}{3^2\times5\times7\times9} \cdots$

Find the general term and interval of convergence of this series.

120. Proposed by JOSIAH H. DRUMMOND, LL. D., Portland, Me.

A hollow sphere has within it a solid sphere; a quantity of water equal to $1/m$ of the capacity of the hollow sphere is poured in and just covers the solid sphere. Prove that there are two solid spheres, either of which answers the conditions; also find the maximum value $1/m$, beyond which the question is not possible.

*** Solutions of these problems should be sent to J. M. Colaw not later than July 10.

CALCULUS.

111. Proposed by G. B. M. ZERR, A.M., Ph.D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

(a). Find the dimensions of a cup, capacity c, in the form of a frustum of a pyramid regular, of n faces, so that its internal surface is a minimum.

(b). Find the dimensions of a cup, capacity c, in the form of a frustum of a hyperbolioid or of a paraboloid, whichever it is, so that its internal surface is a minimum.

*** Solutions of these problems should be sent to J. M. Colaw not later than July 10.

MISCELLANEOUS.

91. Proposed by ARTEMAS MARTIN, LL. D., U. S. Coast and Geodetic Survey Office, Washington, D. C.

The following sides and area are given for a rational triangle in the table of rational scalene triangles on page 167 of Dr. Halsted's "Metrical Geometry" (Boston, 1881), viz.: sides, 21, 61, 65; area, 420. The same sides and area are given in Septimus Tebay's "Mensuration" (London and Cambridge, 1868), in a table on page 113.

The sides of this triangle can not all be correct because they are all *odd*.

Assuming that the *area* given is correct, it is required to determine the error in the sides.

92. Proposed by J. T. COLE, Columbus, Ohio.

A staff a=60 feet high, casting a shadow on a horizontal plane due north b=20 feet long, falls due northeast. Find the area covered by the shadow.

*** Solutions of these problems should be sent to J. M. Colaw not later than July 10.

NOTE ON THE SOLAR ECLIPSE OF MAY 28TH.

This eclipse was witnessed by the editor and his friend, Mr. Abel, from the campus of Tulane University of New Orleans, La. The atmospheric and meteorological conditions for observing this interesting phenomenon was most favorable. In the morning, at sun rise, the eastern sky was skirted with clouds, but by the time the first contact occurred, these had sunk nearer the horizon so as to leave the sun in a perfectly clear sky. First contact occurred at 6:25 A. M.; the second at 7:30; the third at 7:31 minutes 12 seconds, and the fourth at 8:43. Thus the period of totality was 1 minute and 12 seconds. Such a short period of totality did not give much time for the observation of those interesting phenomena attending an eclipse, viz., the "shadow bands" or "fringes," the corona, prominences, etc.

Prof. Ayers of the Tulane University had been making extensive preparations for the event for several months and during the time of the eclipse, took every precaution to eliminate as far as possible all sources of error. To this end it was ordered that visitors should not be allowed to approach the observatory nearer than 100 yards. This precaution was taken, since the motion, caused by the moving about of the visitors on the shallow crust of the earth here floating on a watery bed would be easily transmitted to the observatory and thus effect the instruments used in making observations. Professors Hume, Fulton, and Johnson, of the University of Mississippi, brought their astronomical instruments with them to Tulane University and coöperated in their observations with the Tulane University men.

It was through the courtesy of Prof. Hume that the editor was admitted to the sacred precincts of that eager and enthusiastic band of scientists, and through the courtesy of Professor Johnson was also permitted to take a view of the eclipsed sun with the instrument with which he was working.

The darkness was not as intense as one might suppose and not as dark as on previous occasions. The corona was very beautiful, though perhaps not so beautiful as has been seen during previous eclipses. Instead of streaming out on all sides, radiant filaments, beams, and sheets of pearly light, forming an irregular stellate halo, there appeared this time only a broad band of light lying in the direction in which the moon crossed the sun. The corona, therefore, had a marked resemblance to the corona of 1867. The "shadow bands" were quite clear to be seen at the time of totality. Whether a satisfactory explanation of their origin and nature has been discovered remains to be seen. Mr. Abel made several photographic exposures, but with what success is not yet known. Thorough and detailed accounts of this eclipse together with the results of the numerous observations made throughout the country, of course, will be given in the various Astronomical journals.

May 29, 1900.

EDITORIALS.

Mr. Sylvester Robins, of North Branch, N. J., one of our valued contributors, died suddenly and very unexpectedly, on Wednesday evening, April 25.

Mr. T. M. Putnam, Instructor in Mathematics in the University of Texas, has applied for a leave of absence to accept a Fellowship in the University of Chicago.

Dr. L. E. Dickson has resigned his position as Associate Professor of Mathematics in the University of Texas, to accept a call to the University of Chicago.

The B. F. Johnson Publishing Company, Richmond, Va., have in press a new two-book series of Arithmetics, by J. M. Colaw, Associate Editor of the MONTHLY, and J. K. Ellwood, Principal of the Colfax School, Pittsburg.

Thomas Craig, Ph. D., Professor of Pure Mathematics at Johns Hopkins University, a distinguished and widely-known mathematician, died in Baltimore, on May 8. Death was due to heart trouble. His wife and two daughters survive him. Professor Craig was born in Pittston, Pa., in the year 1855, and was in the very prime of life. He graduated from Lafayette College as a civil engineer in 1875, and took the degree of Doctor of Philosophy at Johns Hopkins in 1878. From his connection with the University, Professor Craig began his work in mathematics, which he continued to the time of his death, having borne a most important part in the work of the mathematical department. For many years he was editor of *The American Journal of Mathematics*. His more important works include "A Treatise on Linear Differential Equations," "A Treatise on Projections," and "Motions of Fluids." At the time of his death he was engaged on a new work entitled "Advanced Theory of Surfaces." Professor Craig was a frequent contributor to the different mathematical journals.

BOOKS AND PERIODICALS.

Rational Elementary Arithmetic. By H. H. Belfield, A. M., Ph. D., Director of the Chicago Manual Training School, and Sarah C. Brooks, Supervisor of Primary Grades, St. Paul, Minn. 268 pages. Price, 45 cents. Chicago: Scott, Foresman and Company. 1899.

This book contains an abundance of concrete work and graphic illustration. It is rich in suggestion and in thought-arousing matter. The book in the hands of an intelligent and skillful teacher should produce excellent results, but we should hesitate before placing it in the hands of an untrained or "unsupervised" teacher. J. M. C.

Bailey-Wiemer Series—First and Second Books in Arithmetic. By F. M. Wiemer, Principal of First District School, Milwaukee, Wis., assisted by M. A.

Bailey, A. M., Professor of Mathematics, Kansas State Normal School. 96 and 176 pages. New York: American Book Company. 1899.

These books abound in figures and indicated operations, but are deficient in objective work. If figure processes were the *end*, instead of the *means* to the end, these little volumes would be especially meritorious. However, with an expert teacher to supply the needed concrete work, they should give good results. J. M. C.

Manual of Experimental Physics. For Secondary Schools. By Fred R. Nichols, Charles H. Smith, and Charles M. Turton, Instructors in Physics in Chicago High School. 8 vo., cloth, 324 pages. Chicago: Ginn & Co.

This book embraces a thorough course in Laboratory work in Physics for High School and Academies. The work aims to make laboratory work inductive as far as possible. B. F. F.

Nine Ninety-Nine Graded Problems in Arithmetic. By Fred V. Lester, A. M., Superintendent of Schools, Ticonderoga, N. Y. 94 pages. Syracuse: C. W. Bardeen. 1899.

This little book gives a serviceable collection of problems for review work. J. M. C.

Lessons in Arithmetic—Primary Number. By C. L. Howard, of the Columbia School, St. Louis, 72 pages. Price, 25 cents. St. Louis: W. S. Bell & Son. 1899.

The author puts forth this small volume as the result of his experience and the growth of trial. It seems to be well suited for use at the stage when children are acquiring their first knowledge of numbers and number relations. J M. C.

Graded Lessons in Arithmetic. Books II—VIII. By Wilbur F. Nichols, A. M., Principal Hamilton Street School, Holyoke, Mass. Price, 25 cents each. Boston: Thompson, Brown & Co. 1897-1899.

The plan of the author is to present the elements of many topics in the lower grades, and then to make the work in each topic more difficult through the subsequent grades. These books furnish a large number of well-graded examples of great variety on all the different topics. An examination in detail reveals many good features. J. M. C.

Gay's Problems in Arithmetic. Books I and II. By George E. Gay, Superintendent of Schools, Malden, Mass. Boston: Benj. H. Sanborn & Co. 1898.

Each of these books contains 1,000 problems designed for written work. The problems are progressive and practical, and are presented in an attractive form. J. M. C.

First Steps in Arithmetic. By Ella M. Pierce, Supervisor of Primary Grades, Public Schools, Providence, R. I. 160 pages. Price, 36 cents. Boston: Silver, Burdett & Co. 1899.

Twenty is the extreme of the numbers taught. The ordinary tables of measurement are freely used. The book is intended for beginners, and is well adapted to its end. J. M. C.

The Werner Arithmetic. Book III. By Frank H. Hall. 282 Pages. Chicago: Werner School Book Company. 1898.

In Books I and II of this series, classification is made subordinate to gradation, but in this book the topical arrangement is given. The number of topics has been wisely reduced. The book is carefully written, and contains practical problems in great variety.

The introduction of the elements of algebra and geometry is to be commended, but the systematic recurrence of these subjects in parts of the text, where they do not serve to illuminate the arithmetical treatment, detracts somewhat from the general excellence of the work. J. M. C.

Rand-McNally Primary Arithmetic. Revised Edition, 1899. By Edwin C. Hewett, LL. D., Ex-President of the Illinois State Normal University. 268 pages. Price, 30 cents. Chicago: Rand, McNally & Co.

This Primany book is unusually full in the early steps, and gives a good preparation for the advanced grades. The directions and suggestions to teachers are full and generally to the point. J. M. C.

Graded Work in Arithmetic. Fifth Book. By S. W. Baird, Principal Franklin Grammar School, Wilkesbarre, Pa. 356 pages. Price, 65 cents. New York: American Book Company. 1899.

This book completes a well-graded series, and is intended for grammar school grades. There are no rules, and a minimum of definitions and explanations. The supply of practical problems is abundant and varied. One chapter is devoted to Algebraic equations. J. M. C.

Lippincott's Arithmetics, Mental, Elementary and Practical. By J. Morgan Rawlins, A. M. Prices, 35 cents, 40 cents, and 65 cents, respectively. Philadelphia: J. M. Lippincott Company. 1899.

The aim of these books is "to prepare students for practical life by the development of thought power, while making them masters of the mechanical processes." The advanced book seems to be overburdened with definitions, principles, analyses, synthesis, rules, etc., and several of the topics included could well have been omitted as having no utilitarian value. The problems are fresh, well-arranged, and adequate in number. The first forty pages of the Elementary book are especially well suited to the needs of the beginner. J. M. C.

The American Monthly Review of Reviews. An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, $2.50 per year in advance. Single numbers, 25 cents. The Review of Reviews Co., New York.

Dr. Albert Shaw describes "Paris and the Exposition of 1900" in his magazine, the *Review of Reviews*, for June. Dr. Shaw regards Paris itself, the typically modern city, as an inseparable part of the great fair. So far from complaining of the incompleteness of the Exposition in the opening month, as many visitors have, Dr. Shaw welcomed the opportunity to see so many of the wonders of the fair in the making. His article is by far the most discriminating estimate of the real value of the Paris show that has been published on this side of the Atlantic.

The Literary Digest. A Weekly Compendium of the Contemporaneous Thought of the World. Price, $3.00 per year in advance. Single numbers, 10 cents. Funk and Wagnalls Co., Publishers, 30 Lafayette Place, New York.

The Cosmopolitan. An International Illustrated Monthly Magazine. Edited and published by John Brisben Walker. Price, $1.00 per year in advance. Single numbers, 10 cents. Irvington-on-the-Hudson.

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ON A DETERMINANT EACH OF WHOSE ELEMENTS IS THE PRODUCT OF k FACTORS.

By PROF. W. H. METZLER, Syracuse University, Syracuse, N. Y.

1. In Muir's *Theory of Determinants*, page 117, the following example (slightly modified) is given without comment :

$$\begin{vmatrix} h_1m_1 & h_1n_1 & k_1x_1 & k_1z_1 \\ h_2m_1 & h_2n_1 & k_2x_1 & k_2z_1 \\ p_1m_2 & p_1n_2 & q_1x_2 & q_1z_2 \\ p_2m_2 & p_2n_2 & q_2x_2 & q_2z_2 \end{vmatrix} = \begin{vmatrix} h_1 & k_1 & 0 & 0 \\ h_2 & k_2 & 0 & 0 \\ 0 & 0 & p_1 & q_1 \\ 0 & 0 & p_2 & q_2 \end{vmatrix} . \begin{vmatrix} m_1 & 0 & m_2 & 0 \\ n_1 & 0 & n_2 & 0 \\ 0 & x_1 & 0 & x_2 \\ 0 & z_1 & 0 & z_2 \end{vmatrix}$$

$$= \begin{vmatrix} h_1k_1 \\ h_2k_2 \end{vmatrix} . \begin{vmatrix} p_1q_1 \\ p_2q_2 \end{vmatrix} . \begin{vmatrix} m_1n_1 \\ m_2n_2 \end{vmatrix} . \begin{vmatrix} x_1z_1 \\ x_2z_2 \end{vmatrix} .$$

From this it is seen that if we take n_1 determinants

$$| b_{1n_2}^{(n_2, j_1)} | , (j_1=1, 2 \ldots . n_1),$$

of order n_2 and form a determinant of order $n=n_1n_2$ by arranging them along the principal diagonal as in the first factor of the right-hand member of the first equation in the above example, all the other elements being zeros, and n_2 determinants

$$| b_{1n_1}^{(n_1, j_2)} | , (j_2=1, 2 \ldots . n_2),$$

of order n_1 and form another determinant of order n in which the elements in the $(\alpha n_2+\beta)$th column of the βth n_1 rows are the elements of the $(\alpha+1)$th column of

$$| b_{1n_1}^{(n_1,\beta)} | , (\alpha=0, 1 \ldots . n-1), (\beta=1, 2 \ldots . n_2),$$

all the other elements being zeros, the product of these two determinants of the nth order is a determinant of the nth order whose elements are products of two factors.

It is also seen that the elements of $| b_{1n_2}^{(n_2,j_1)} |$ are found in all the columns, but in the j_1th n_2 rows only of the product, and that the elements of $| b_{1n_1}^{(n_1,j_2)} |$ are found in all the rows, but in the (αn_2+j_2)th columns only of the product.

2. We may now form two determinants of the nth order $(n=n_1n_2n_3)$ precisely as in Art. 1, first by taking n_3 determinants of order n_1n_2 each of whose elements is the product of two factors as there found, and second by taking n_1n_2 determinants

$$| b_{1n_3}^{(n_3,j_3)} | , (j_3=1, 2 \ldots . (n/n_3)$$

of order n_3, and the product of these two determinants of the nth order will be a determinant of the nth order each of whose elements is the product of three factors, one factor an element from a determinant of each of the three orders.

Continuing this process we arrive at a determinant of the order $n=n_1n_2 \ldots n_k$, each of whose constituents is the product of k factors, one factor an element from a determinant of each of the k given orders $n_1, n_2, \ldots . n_k$

3. Let $\left\{ \frac{\alpha}{\beta} \right\}$ denote the greatest integer in $\frac{\alpha}{\beta}$, $R\frac{\alpha}{\beta}$ denote the remainder on dividing α by β,

$$| b_{1n_g}^{(n_g,j_g)} | , (g=1, 2 \ldots . k), (j_g=1, 2 \ldots n/n_g),$$

denote the n/n_g determinants of order n_g. Then if $n=n_1n_2 \ldots . n_k$ we have the theorem

$$A=| a_{1n} | =\Pi | b_{1n_g}^{(n_g,j_g)} | , (g=1, 2 \ldots . k), (j_g=1, 2 \ldots . n/n_g) \ldots . (1).$$

Where the element in the xth row and yth column of A,

$$a_{xy}=b^{\left(n_1, \left\{ \frac{x-1}{n_1} \right\}+1\right)}_{R\frac{x-1}{n_1}+1, \left\{ \frac{y-1}{n/n_1} \right\}+1}$$